

INFLUENCE FUNCTIONS, FOLLOWERS, and COMMAND GAMES

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- ▶ Few studies have been devoted to models of **influence**:
 - ▶ the *Hoede-Bakker index*
 - ▶ the *command games of Hu and Shapley*
- ▶ *We propose an influence model, starting from the Hoede-Bakker index, and we compare it to the command games model*

Basic material

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- ▶ In set notation, if i corresponds to S , we denote Bi by **$B(S)$ (set of voters voting finally YES)**, with $B(S) \subseteq N$
- ▶ Hence $B : 2^N \rightarrow 2^N$, and the cardinality of $\mathcal{B}$ is $(2^n)^{(2^n)} = 2^{n2^n}$

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 - ▶ If the parents have the same inclination, the child follows the parents
 - ▶ Otherwise, the child follows his own inclination
 - ▶ No influence between father and mother
- ▶ The influence function is:

i	$(1, 1, 1)$	$(1, 1, -1)$	$(1, -1, 1)$	$(-1, 1, 1)$	$(1, -1, -1)$	$(-1, 1, -1)$	$(-1, -1, 1)$	$(-1, -1, -1)$
$B(i)$	$(1, 1, 1)$	$(1, 1, -1)$	$(1, -1, 1)$	$(1, 1, 1)$	$(-1, -1, -1)$	$(-1, 1, -1)$	$(-1, -1, 1)$	$(-1, -1, -1)$

or in set notation:

S	123	12	13	23	1	2	3	$\emptyset$
$B(S)$	123	12	13	123	$\emptyset$	2	3	$\emptyset$

Typical examples of influence functions

► *The majority influence function* $\text{Maj}^{[t]}$

Let $n \geq t > \lfloor \frac{n}{2} \rfloor$, $i^+ := \{k \in N \mid i_k = +1\}$

$$\text{Maj}^{[t]}_i := \begin{cases} 1_N, & \text{if } |i^+| \geq t \\ -1_N, & \text{if } |i^+| < t \end{cases}, \quad \forall i \in I.$$

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- Let $t \in (0, n]$ and $\epsilon \in \{-1, +1\}$. Define for any $i \in I$ the set $i^\epsilon := \{k \in N \mid i_k = \epsilon\}$. Functions B satisfying if $|i^\epsilon| \geq t$, then $(Bi)^\epsilon \supseteq i^\epsilon$ are called *mass psychology influence functions*

Influence indices (1/2)

- Some notations: let $S \subseteq N$, $S \neq \emptyset$, and $j \in N \setminus S$.

$I_S = \{i \in I \mid \forall j, k \in S, i_j = i_k\}$ (**unanimity in S**). For any $i \in I_S$, we put $i_S := i_j$ for any $j \in S$

$I_{S \rightarrow j} := \{i \in I_S \mid i_j = -i_S\}$ (**potential influence of S on j**)

$I_{S \rightarrow j}^*(B) := \{i \in I_{S \rightarrow j} \mid (Bi)_j = i_S\}$ (**observed influence of S on j**)

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- Example: $n = 5$, $S = 123$, $j = 4$. Then

$I_{123} =$
 $\{(\textcolor{red}{11111}), (\textcolor{red}{111} - 11), (\textcolor{red}{1111} - 1), (\textcolor{red}{111} - 1 - 1), +\text{idem with } -1 - 1 - 1\}$

$I_{123 \rightarrow 4} = \{(\textcolor{red}{111} - \textcolor{blue}{11}), (\textcolor{red}{111} - \textcolor{blue}{1} - 1), (-\textcolor{red}{1} - \textcolor{red}{1} - \textcolor{blue}{111}), (-\textcolor{red}{1} - \textcolor{red}{1} - \textcolor{blue}{11} - 1)\}$

Supposing $B(\textcolor{red}{111} - \textcolor{blue}{11}) = \textcolor{red}{1} - \textcolor{red}{111} - 1$, $B(\textcolor{red}{111} - \textcolor{blue}{1} - 1) = -\textcolor{red}{11} - \textcolor{blue}{11} - 1$,
 $B(-\textcolor{red}{1} - \textcolor{red}{1} - \textcolor{blue}{111}) = -\textcolor{red}{1} - \textcolor{red}{1111}$, and $B(-\textcolor{red}{1} - \textcolor{red}{1} - \textcolor{blue}{11} - 1) = \textcolor{red}{1} - \textcolor{red}{1} - \textcolor{blue}{111}$,
 we have $I_{123 \rightarrow 4}^*(B) = \{(\textcolor{red}{111} - \textcolor{blue}{11}), (\textcolor{red}{111} - \textcolor{blue}{1} - 1)\}$

Influence indices (2/2)

- ▶ The *possibility influence index of coalition S on player j* is defined as

$$\bar{d}(B, S \rightarrow j) := \frac{|I_{S \rightarrow j}^*(B)|}{|I_{S \rightarrow j}|}$$

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- ▶ The *weighted influence index of coalition S on player j* is given by

$$d_\alpha(B, S \rightarrow j) := \frac{\sum_{i \in I_{S \rightarrow j}^*(B)} \alpha_i^{S \rightarrow j}}{\sum_{i \in I_{S \rightarrow j}} \alpha_i^{S \rightarrow j}}$$

with $\alpha_i^{S \rightarrow j}$, $i \in I_S$, are weights in $[0, 1]$.

Examples (1/2)

- The family example (1=child, 2=mother, 3=father):

$$d_{\alpha}(B, 1 \rightarrow 2) = d_{\alpha}(B, 1 \rightarrow 3) = d_{\alpha}(B, 2 \rightarrow 3) = d_{\alpha}(B, 3 \rightarrow 2) = 0$$

$$d_{\alpha}(B, 12 \rightarrow 3) = d_{\alpha}(B, 13 \rightarrow 2) = 0, \quad d_{\alpha}(B, 23 \rightarrow 1) = 1$$

$$\bar{d}(B, 2 \rightarrow 1) = \bar{d}(B, 3 \rightarrow 1) = \frac{1}{2}$$

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$$d^*(B, 2 \rightarrow 1) = d^*(B, 3 \rightarrow 1) = \frac{1}{4}$$

Examples (2/2)

- The majority function $\text{Maj}^{[t]}$:

$$d_{\alpha}(\text{Maj}^{[t]}, S \rightarrow j) = \begin{cases} 1, & \text{if } s \geq t \\ \frac{\sum_{i \in I_{S \rightarrow j}^{+}, \geq t} \alpha_i^{S \rightarrow j} + \sum_{i \in I_{S \rightarrow j}^{-}, < t} \alpha_i^{S \rightarrow j}}{\sum_{i \in I_{S \rightarrow j}} \alpha_i^{S \rightarrow j}}, & \text{if } s < t. \end{cases}$$

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- ▶ For the identity function Id , $d_{\alpha}(\text{Id}, S \rightarrow j) = 0$.
- ▶ For the reversal function $-\text{Id}$, $d_{\alpha}(-\text{Id}, S \rightarrow j) = 1$.

Followers and kernel

- ▶ $F_B : 2^N \rightarrow 2^N$ *the follower function of B*
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Proposition

Let $B \in \mathcal{B}$. Then the following holds:

1. Whenever $S \cap T = \emptyset$, $F_B(S) \cap F_B(T) = \emptyset$.
2. F_B is an isotone function ($S \subseteq S'$ implies $F_B(S) \subseteq F_B(S')$).
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- ▶ The *kernel* of B is the set of “truly” influential coalitions:

$$\mathcal{K}(B) := \{S \in 2^N \mid F_B(S) \neq \emptyset, \text{ and } S' \subset S \Rightarrow F_B(S') = \emptyset\}$$

Examples

- ▶ The family example (1=child, 2=mother, 3=father)

$$F_B(\emptyset) = F_B(1) = \emptyset, \quad F_B(2) = \{2\}, \quad F_B(3) = \{3\}$$

$$F_B(12) = \{1, 2\}, \quad F_B(13) = \{1, 3\}, \quad F_B(23) = \{1, 2, 3\}$$

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- ▶ Hence, there are potentially $(2^n)^{(2^n)}$ follower functions, **but any function $2^N \rightarrow 2^N$ is not necessarily a follower function corresponding to some influence function B .**
- ▶ Consequently, the mapping $\Phi : \mathcal{B} \rightarrow (2^N)^{(2^N)}$, defined by $B \mapsto \Phi(B) := F_B$ is **neither a surjection nor an injection**. We put $\Phi(\mathcal{B}) =: \mathcal{F}$ the set of all follower functions.

Relations between $\mathcal{F}$ and $\mathcal{B}$

The following natural questions may be raised:

- ▶ Given a function $F : 2^N \rightarrow 2^N$, which are the sufficient and necessary conditions so that there exists $B \in \mathcal{B}$ such that F is the follower function of B , i.e., $F = F_B$?

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- ▶ If $F : 2^N \rightarrow 2^N$ is indeed a follower function, can we easily find examples of B 's such that $F_B = F$?
- ▶ Moreover, can we find $\Phi^{-1}(F)$, i.e., the set of all influence functions that lead to the follower function F ? What is the (algebraic) structure of $\Phi^{-1}(F)$?

When F is a follower function?

Proposition

A function $F : 2^N \rightarrow 2^N$ belongs to $\mathcal{F}$ if and only if

1. $F(\emptyset) = \emptyset$;
2. F is isotone, i.e., $S \subseteq T$ implies $F(S) \subseteq F(T)$;
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Moreover, the smallest and greatest influence functions belonging to $\Phi^{-1}(F)$ are respectively $\underline{B}_F$ and $\overline{B}_F$, defined by, in set notation:

$$\underline{B}_F(S) = F(S), \quad \overline{B}_F(S) = \overline{F(\overline{S})}.$$

We call these influence functions the *lower and upper inverses of F*

Examples

Example

Consider $F \equiv \emptyset$. The reversal function $-\text{Id}$, defined by $-\text{Id}i := -i$, for each $i \in I$ is an inverse of F . The lower inverse is the constant function $B \equiv -1_N$, while the upper inverse is $B \equiv 1_N$. Evidently, $\Phi^{-1}(\emptyset) \neq [-1_N, 1_N] = \mathcal{B}$.

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Example

Consider $F = \text{Id}$. The lower and upper inverses collapse to Id . Hence, $\Phi^{-1}(\text{Id}) = \{\text{Id}\}$.

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Example

Let $n = 3$ and the following function F be defined as follows:

S	$\emptyset$	1	2	3	12	13	23	123
$F(S)$	$\emptyset$	$\emptyset$	2	$\emptyset$	2	3	12	123

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The upper and lower inverses are

i	$\emptyset$	1	2	3	12	13	23	123
$\bar{B}_F i$	$\emptyset$	3	12	13	123	13	123	123
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Proposition

The function Φ satisfies the following properties:

1. For any $B \in \mathcal{B}$, for any $S \subseteq N$, $\Phi(B)(S) \subseteq B(S)$ (or more compactly $\Phi(B) \leq B$).
2. The set of fixed points of Φ (i.e., for which $\Phi(B) = B$) is exactly $\mathcal{F}$. Hence, $\Phi^2 = \Phi^3 = \dots = \Phi$.

The structure of $\Phi^{-1}(F)$ (1/2)

- ▶ With the usual ordering $\leq$ of functions, $(\Phi^{-1}(F), \leq)$ is a poset, which is a subset of the Boolean lattice $[\underline{B}_F, \overline{B}_F]$

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- ▶ Hence, an element of $[\underline{B}_F, \overline{B}_F]$ is denoted by the 2^n -dim vector $(T_\emptyset, \dots, T_N)$, where $T_S \subseteq D_S$ for each $S \subseteq N$. With this notation, $\underline{B}_F$ and $\overline{B}_F$ are denoted by $(\emptyset, \dots, \emptyset)$ and $(D_\emptyset, \dots, D_N)$ respectively.

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Example

S	$\emptyset$	1	2	3	12	13	23	123
$F(S)$	$\emptyset$	$\emptyset$	2	$\emptyset$	2	3	12	123
$\overline{B}_F(S)$	$\emptyset$	3	12	13	123	13	123	123
$\underline{B}_F(S)$	$\emptyset$	$\emptyset$	2	$\emptyset$	2	3	12	123
D_S	$\emptyset$	3	1	13	13	1	3	$\emptyset$

The structure of $\Phi^{-1}(F)$ (2/2)

Theorem

For any $F \in \mathcal{F}$, $(\Phi^{-1}(F), \leq)$ has the following properties:

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3. $\Phi^{-1}(F)$ is autodual. The duality is expressed as follows: to each element $B := (T_\emptyset, \dots, T_S, \dots, T_N)$ of $\Phi^{-1}(F)$ corresponds the element $B' := (D_N \setminus T_N, \dots, D_{\overline{S}} \setminus T_{\overline{S}}, \dots, D_\emptyset \setminus T_\emptyset)$.

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4. There are $\sum_{S \subseteq N} |D_S|$ join-irreducible elements, one for each $k \in D_S$, $S \subseteq \overline{N}$, either of the form $(k_S \emptyset)$ if this element belongs to $\Phi^{-1}(F)$, otherwise of the form $(k_S k_{\overline{S}} \emptyset)$, where the notation $(k_S \emptyset)$ is a shorthand for $(\emptyset, \dots, \emptyset, k, \emptyset, \dots, \emptyset)$, where k is at position S , and similarly for $(k_S k_{\overline{S}} \emptyset)$.

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5. The lattice is distributive, hence ranked, and its height is $h = \sum_{S \subseteq N} |D_S|$.

Example

S	$\emptyset$	1	2	3	12	13	23	123
D_S	$\emptyset$	3	1	13	13	1	3	$\emptyset$

- ▶ For $S = 1, k = 3$: $(3_1\emptyset)$ belongs to $\Phi^{-1}(F)$, so it is join-irreducible
- ▶ For $S = 2, k = 1$: $(1_2\emptyset)$ belongs to $\Phi^{-1}(F)$, so it is join-irreducible
- ▶ For $S = 3, k = 1$: $(1_3\emptyset)$ belongs to $\Phi^{-1}(F)$, so it is join-irreducible
- ▶ For $S = 3, k = 3$: $(3_3\emptyset)$ belongs to $\Phi^{-1}(F)$, so it is join-irreducible
- ▶ For $S = 12, k = 1$: $(1_{12}\emptyset)$ does not belong to $\Phi^{-1}(F)$, so $(1_{12}1_3\emptyset)$ is join-irreducible
- ▶ For $S = 12, k = 3$: $(3_{12}\emptyset)$ does not belong to $\Phi^{-1}(F)$, so $(3_{12}3_3\emptyset)$ is join-irreducible
- ▶ For $S = 13, k = 1$: $(1_{13}\emptyset)$ does not belong to $\Phi^{-1}(F)$, so $(1_{13}1_2\emptyset)$ is join-irreducible
- ▶ For $S = 23, k = 3$: $(3_{23}\emptyset)$ does not belong to $\Phi^{-1}(F)$, so $(3_{23}3_1\emptyset)$ is join-irreducible

The command games of Hu and Shapley (2003) (1/2)

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- ▶ *Minimal boss sets* and *minimal approval sets* for k :

$$Boss_k^* := \{S \in Boss_k \mid S' \subset S \Rightarrow S' \notin Boss_k\}$$

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 $\omega(S)$ is the *set of all members that are 'commandable' by S*

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- ▶ $\omega(\emptyset) = \emptyset$, $\omega(N) = N$, $\omega(S) \subseteq \omega(S')$ whenever $S \subset S'$.

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- ▶ $N = \{1 = \text{king}, 2 = \text{man}, 3 = \text{wife}, 4 = \text{child}\}$

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► Boss sets and approval set are:

$$\text{Boss}_1^* = \text{Boss}_1 = \emptyset, \quad \text{App}_1^* = \text{App}_1 = \{234\}$$

$$\text{Boss}_2^* = \{1\}, \quad \text{Boss}_2 = \{1, 13, 14, 134\}$$

$$\text{Boss}_3^* = \text{Boss}_4^* = \{2\}, \quad \text{Boss}_3 = \{2, 12, 24, 124\}, \quad \text{Boss}_4 = \{2, 12, 23, 123\}$$

$$\text{App}_k = \text{App}_k^* = \emptyset \text{ for } k = 2, 3, 4.$$

Example: the Confucian model of society (2/2)

- The command function is

$$\begin{aligned}\omega(1) &= \omega(13) = \omega(14) = \omega(134) = \{2\} \\ \omega(2) &= \omega(23) = \omega(24) = \omega(234) = \{3, 4\} \\ \omega(3) &= \omega(4) = \omega(34) = \emptyset \\ \omega(12) &= \omega(123) = \omega(124) = \{2, 3, 4\} \\ \omega(N) &= N\end{aligned}$$

Command games and command functions (1/3)

- ▶ A set of command games $(N, \mathcal{W}_k)$, $k \in N$, can be viewed as a mapping $\Omega : N \times 2^N \rightarrow \{0, 1\}$

$$(k, S) \mapsto \Omega(k, S) = \begin{cases} 1, & \text{if } S \in \mathcal{W}_k \\ 0, & \text{else} \end{cases}$$

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- ▶ In addition we propose that $S_1 \cap \dots \cap S_l \neq \emptyset$ for each $\mathcal{W}_k$. This implies in particular that there are no two disjoint boss sets.

Command games and command functions (2/3)

Definition

A *normal command game* Ω is a set of simple games $\{(N, \mathcal{W}_k), k \in N\}$ satisfying the two conditions:

1. For each $k \in N$, there exists a minimal nonempty family of nonempty subsets $S_1^k, \dots, S_{l_k}^k$ (called the *generating family of $\mathcal{W}_k$*) such that $\mathcal{W}_k = \uparrow S_1^k \cup \dots \cup \uparrow S_{l_k}^k$.
2. For each $k \in N$, $S_1^k \cap \dots \cap S_{l_k}^k \neq \emptyset$.

We denote by $\mathcal{G}$ the set of all normal command games (viewed as a subset of $2^{N \times 2^N}$).

The Confucian example (ctd)

- In the Confucian example, we had:

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- Hence:

$$\{S_1^1, \dots, S_{l_1}^1\} = \{1234\}$$

$$\{S_1^2, \dots, S_{l_2}^2\} = \{1\}$$

$$\{S_1^3, \dots, S_{l_3}^3\} = \{S_1^4, \dots, S_{l_4}^4\} = \{2\}$$

.

which shows that it is a normal command game

Command games and command functions (3/3)

Given a function ω from 2^N to 2^N , what are the sufficient and necessary conditions so that it is the command function of a normal command game? The following proposition answers this question.

Proposition

Let $\omega \in (2^N)^{(2^N)}$. Then ω corresponds to some normal command game if and only if the following conditions are satisfied:

1. $\omega(\emptyset) = \emptyset$, $\omega(N) = N$;
2. ω is isotone, i.e. it is monotone w.r.t. set inclusion;
3. If $S \cap S' = \emptyset$, then $\omega(S) \cap \omega(S') = \emptyset$.

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Theorem

1. *Let B be an influence function. Then there exists a unique normal command game Ω equivalent to B if and only if $F_B(N) = N$.*
2. *Let Ω be a normal command game. Then any influence function in $\Phi^{-1}(\omega)$ is equivalent to Ω , in particular the upper inverse $\overline{B}_\omega$ and the lower inverse $\underline{B}_\omega$.*

The Confucian example (end)

The upper and lower inverses of ω are

S	$\emptyset$	1	2	3	4	12	13	14
$\overline{B}_\omega(S)$	$\emptyset$	12	134	1	1	N	12	12
$\underline{B}_\omega(S)$	$\emptyset$	2	34	$\emptyset$	$\emptyset$	234	2	2
S	23	24	34	123	124	134	234	N
$\overline{B}_\omega(S)$	134	134	1	N	N	12	134	N
$\underline{B}_\omega(S)$	34	34	$\emptyset$	234	234	2	34	N