

Using Approval Balloting in Multi-Winner Elections

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Outline

- Objective: Survey methods of conducting multi-winner elections using approval balloting
- All procedures are anonymous and neutral
- Set of potentially winning subsets of candidates is arbitrary, treated as parameter
- Three classes of procedures:
 - Scoring
 - Threshold
 - Centralization

Approval Balloting

- Approval Voting is a well-known procedure for single-winner elections
- Voters vote for (approve of) any number of candidates; the winner is the most-approved candidate(s)
- Approval balloting refers to use of approval ballots: Each candidate is approved, or not, by each voter
- Approval balloting is natural for committee elections since each ballot, and the outcome of the election, are subsets of the set of all candidates

Notation

- There are $n > 1$ voters
- There are $m > 1$ candidates
- The set of all candidates is $[m] = \{1, 2, \dots, m\}$
- Voter i 's ballot is $V_i \in 2^{[m]}$
- In general, the set of possible committees is $2^{[m]} - \emptyset$
- In any particular election, the set of *admissible* committees is $A \subseteq 2^{[m]} - \emptyset$
- Examples:
 - $A_k = \{S \in 2^{[m]} : |S| = k\}$
 - $A_F = 2^{[m]} - \emptyset$

Examples

Example 1

Voter	1	2	3	4
Ballot	1	12	13	14

$$A = A_F$$

Example 2

Voter	1	2	3	4	5	6	7	8	9
Ballot	2	12	12	13	37	45	46	47	48

$$A = A_3$$

Scoring Rules

- Each admissible subset, S , is scored in a way that measures the similarity of S to the voters' ballots, $V_1, V_2, \dots, V_n$
- Note: Score increases with greater similarity
- Winning subset(s): Highest score

Maximum Approval Rule

- Score $App(S) = \sum_i |V_i \cap S|$
- For each voter, similarity = amount of overlap
- Natural extension of approval voting for single-winner elections
- Easy to calculate: $App(S) = \sum_{j \in S} App(j)$
- Easy to apply for $A = A_k$
- Low computational effort for small numbers of candidates
- Tends to choose largest possible committee, so works poorly for $A = A_F$

Reversible Approval Rule

- Score $RApp(S) = \sum_i |(V_i \cap S) \cup (V_i^c \cap S^c)|$
- For each voter, similarity = amount of overlap of winners with voted-for candidates and of losers with voted-against candidates
- “Corrects” bias of Maximum Approval toward larger subsets -- natural extension if A includes subsets of different sizes
- Easy to calculate: $RApp(S) = \sum_{j \in S} App(j) - Opp(j)$
- Easy to apply for $A = A_F$
- Generally, results similar to Maximum Approval for $A = A_k$

Proportional Approval Rule

- To represent more voters by downweighting those with more representatives, Simmons suggest a scoring rule with

$$PApp(S) = \sum_i r(|V_i \cap S|)$$

- The specific score sequence suggested by Simmons is

$$r(k) = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{k}$$

- Note that $r(k) = k$ gives Maximum Approval rule
- Proportional Reversible Approval can be defined using similar principles
- Proportional Approval rules are computationally demanding even with small numbers of candidates

Sequential Proportional Approval Rule

- Suggested by Thiele (c. 1900) for committees of k members ($A = A_k$)
- Elect members sequentially, downweighting voters who are already represented on committee
- Suppose $A = A_1$, and voter i has weight w_i
- Then $WApp(j) = \sum_i w_i |V_i \cap j|$
- Initially, set $w_i = 1$ for all voters i , and $C_0 = [m]$. Select j_1 as candidate maximizing $WApp(j)$. Now set $C_1 = C_0 - j_1$
- If $1 < h \leq k$ and $j_1, j_2, \dots, j_{h-1}$ have been seated, and the remaining candidates are C_{h-1} , reweight the voters

$$w_i^h = \frac{1}{1 + |V_i \cap \{j_1, j_2, \dots, j_{h-1}\}|}$$

- Select j_h as candidate maximizing $WApp(j)$. Now set $C_h = C_{h-1} - j_h$

Satisfaction Approval Rule

- Proposed by Brams and Kilgour (2009); reflects a different approach to “similarity”
- Scoring by
$$Sat(S) = \sum_i \frac{|V_i \cap S|}{|V_i|}$$
- A voter’s satisfaction with S reflects number of candidates in S supported by voter; voter is then motivated not to be too “scattered” in support

Threshold Rules

- Suggested by Fishburn and Pekeč (2004)
- Each admissible subset, S , is scored 1 if it meets some standard of representativeness with respect to the voters' ballots, $V_1, V_2, \dots, V_n$ and 0 otherwise
- Note: A 0-1 system. Score does not increase with greater similarity
- Winning subset(s): Highest score

Mechanism of Threshold Rules

- A *threshold function* is $t: A \rightarrow R^+$; for each admissible committee S , $t(S)$ is the minimum overlap of S with V_i for S to be representative of voter i 's views.
- The Threshold Approval Rule is to select any admissible set S that maximizes $|\{i : |V_i \cap S| \geq t(S)\}|$
- For a Threshold Approval Rule to be neutral, the threshold must be *cardinal*, i.e., $t(S)$ must depend only on $|S|$

Examples of Threshold Rules

- Constant threshold (suitable for $A = A_k$):
 - $t(S) = 1$ can be represented as the proportional approval sequence $r(k) = 1, 1, 1, \dots$
 - $t(S) = 2$, represented as $r(k) = 0, 1, 1, \dots$
- Non-decreasing thresholds (suitable for $A = A_F$):
 - *Majority* threshold: $t(S) = |S|/2$
 - *Strict Majority* threshold $t(S) = (|S| + 1)/2$
- Threshold systems are NP-complete, but computational effort is not excessive if the number of candidate is small

Centralization Systems

- Original reference BKS (2004)
- Any subset of $2^{[m]}$ can be represented as a vertex of the m -dimensional hypercube
- The Hamming distance between two such vertices corresponds to the distance $d(S, T) = |(S \cap T^c) \cup (S^c \cap T)|$ between subsets S and T
- Objective of centralization methods: The “best” committee S is “closest” to $V_1, V_2, \dots, V_n$

Minisum Rule

- Select any $S \in A$ that minimizes $d(S, V) = \sum_i d(S, V_i)$
- Minisum Rule is a Candidate-by-Candidate Majority system for choosing a subset of 2^m
 - The winning subset must include any candidate who is supported on more than half the ballots, and cannot include any candidate who is supported on less than half of the ballots
- Candidate-by-Candidate Majority does not respect admissibility (for example, winning subset may be empty)
- But the Minisum Rule satisfies all of our requirements and is relatively easy to calculate

Minimax Rule

- BKS suggested that a more inclusive committee should be as close as possible to each voter's preferences
- Select any $S \in A$ that minimizes $\max_i d(S, V_i)$
- The Minimax Rule
 - Completely insensitive to clones – adding any number of copies of any voter does not change the outcome
 - Sensitive to extreme voters
 - Computationally difficult.

Weighted Minimax Rules

- KBS (2006) suggested that weighting would make Minimax less sensitive to extremes and more sensitive to duplication of opinion
- Weighting must be applied to the ballots that are cast rather than the voters who cast them
- Distinct ballots $W = \{W_1, W_2, \dots, W_l\}$
- If ballot W_h has weight w_h , then the Weighted Minimax Rule is to select any $S \in A$ that minimizes $\max_h w_h d(S, W_h)$

Weights for Minimax Rules

- Count Weights: $w_h = n_h$ = number of voters who cast ballot W_h
 - Give added weight to duplicate opinions
- Proximity Weights:
$$w_h = \frac{n_h}{\sum_{r=1}^l n_r d(W_h, W_r)}$$
 - Denominator = Total distance from ballot W_h to all other ballots, weighted by number of times the other ballot was cast.
 - Give added weight to duplicate opinions (numerator) and reduced weight to extreme opinions (denominator)

Comparison of Electoral Systems

- This paper has said little about the relative merits of electoral systems. How can they be compared?
 - In the Abstract: Based on their theoretical properties
 - Computationally: Based on the effort required to implement them (in practical cases)
 - Empirically: Based on experience in practice (Example: BKS (2007) contains analysis of complete ballot data from 2003 election to the Council of the Game Theory Society of 12 out of 24 candidates; there were 161 ballots cast)